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Dirac delta representation by exact parametric equations. Application to impulsive vibration systems

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Abstract

A pair of closed parametric equations are proposed to represent the Heaviside unit step function. Differentiating the step equations results in two additional parametric equations, that are also hereby proposed, to represent the Dirac delta function. These equations are expressed in algebraic terms and are handled by means of elementary algebra and elementary calculus. The proposed delta representation complies exactly with the values of the definition. It complies also with the sifting property and the requisite unit area and its Laplace transform coincides with the most general form given in the tables. Furthermore, it leads to a very simple method of solution of impulsive vibrating systems either linear or belonging to a large class of nonlinear problems. Two example solutions are presented.

1. Introduction

There are textbooks, mostly of pure mathematics, that avoid defining the Dirac delta function directly in terms of its numerical values. Instead they tend to define it in terms of both a derivative and an integral. The derivative is that of the Heaviside unit step, the integral is the fundamental or sifting property. However, they also represent it by the most diverse approximations, i.e., limits of functions of either an ε that tends to zero or an n, which tends to infinity. These functions are either algebraic, exponential or transcendental or an Airy function or the Bessel function of the first kind or a Laguerre polynomial [1], there are the Gaussian, the Lorentzian and Kronecker representations [2] and others [3]. Some books state that those limits do not exist [4]. The quest for a better representation seems to go on [5,6]. In most sources, they point out that the delta function is not really a function and even Dirac referred to it as an "improper function" [7].

On the other hand, textbooks oriented to applications tend to define the Dirac delta as zero everywhere except for a single value of the independent variable, where it takes the value of infinity and, furthermore, it has a unit area. Nonetheless, they present the Dirac delta in an approximate form as a Fourier series or trigonometric expansion (and a wavy plot) [8], or as the final product of a limiting process, i.e., a rectangle growing ever taller and thinner with width equal to ε and height equal to $1/\varepsilon$ or a triangle likewise growing ever taller and thinner with base 2ε and height $1/\varepsilon$ with ε tending to a value of zero but each maintaining a unit area

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Nomenclature

a	a constant
$\hat{F}$	impulsive force
$h(x, a)$	riserless Heaviside unit step at $x = a$
$H(x, a)$	parametric Heaviside unit step (with a riser) at $x = a$
I	impulse
L	Laplace transform or length of pendulum according to context
m	mass
P	constant, see Eq. (37)
s	Laplace transform variable
t	time
u	parameter of the parametric representation. Length of the parametric unit step (including the riser) measured from the origin
y	displacement
Y	dimensionless displacement
x	dimensionless independent variable
β	t/τ
$\delta(x, a)$	Dirac delta at $x = a$
λ	y/Y
τ	dimensionless time
$(\ )_i$	referring to impulse instant
$(\ )_p$	referring to post-impulse time
$(\ )_0$	specific value
$(\dot{\ })$	time derivative

throughout the process, sometimes it is an elongated curve, defined only graphically, with vertical symmetry, that goes up from zero ordinate to a non-specified height and returns to zero [9]. Textbooks of either theoretical or applied emphasis state that the Dirac delta function is the derivative of the Heaviside unit step but no differentiation is carried out. Then, after exhibiting a few examples involving the delta (at times with a solution obtained with a somewhat unique reasoning [10]), often point out that in reality such a function does not exist, for example, [4,9]. This is sometimes followed by the statement that the delta is a distribution. This is an unsettling situation, particularly for the typical engineer who is not familiar with the theory of distributions. Fortunately, there is the Laplace transform, which is particularly simple in the case of the Dirac delta and works admirably well, but, unfortunately, it is restricted to linear problems.

In this work, an entirely different approach is used: a pair of parametric equations to represent the Dirac delta are proposed. They are the result of an actual differentiation of another pair of parametric equations, also proposed herein, to represent the Heaviside unit step. The proposed equations are expressed in common algebraic terms, they are closed, exact and they do not require any inequalities. The delta equations have the same function values as those specified in the definition, the area involved has a unit value, they comply with the sifting property and yield the correct Laplace transform. In the solution of differential equations they are handled by elementary calculus and algebra without resorting to any limiting process. Another advantage of the proposed representation is that infinite and infinitesimal quantities can be dealt with indirectly.

As a very natural outcome of this parametric representation, a very simple procedure for solving impulsive problems is developed. The applications refer both to linear problems and a large class of nonlinear problems. The procedure, furthermore, reveals which system components participate in the dynamic process that takes place during the impulse instant, and which do not.

2. The "riserless" Heaviside unit step

The Heaviside unit step function is usually defined as follows:

$$\begin{aligned} h(x,a) &= 0, \quad x<a, \\ h(x,a) &= 1, \quad x>a. \end{aligned} \tag{1}$$

The following alternative single equations may be considered to represent the same function:

$$h(x,a) = \frac{1}{2}\left[1 + \frac{x-a}{\sqrt{(x-a)^2}}\right], \tag{2a}$$

$$h(x,a)=\frac{1}{2}\left[1+\frac{\sqrt{(x-a)^2}}{x-a}\right]. \tag{2b}$$

Long before Heaviside, Cauchy used the same step concept, which he called the ‘‘coefficient limitateur’’ and defined it by Eq. (2a) [11]. This representation has been recently used to advantage in quite diverse research work [12–16].

Fig. 1a is a graphical representation of either Eq. (1) or (2). The reason for using a lower case h in Eqs. (1) and (2) rather than an upper case H used conventionally will be explained further on.

Note that Eq. (1) is made up of four relations, namely, two equations plus two inequalities, while *any one* of the two forms of Eq. (2) is a single equation which contains the same information as all four Eq. (1), in this sense it is ‘‘self-contained’’. The following two equations are differentiations performed on each of the two forms of Eq. (2) in an attempt to see whether either one will yield a similarly self-contained equation for the Dirac delta. Eq. (3a) is the derivative of Eqs. (2a) and (3b) is the derivative of Eq. (2b):

$$\frac{\mathrm{d}h(x,a)}{\mathrm{d}x}=\frac{1}{2}\left[\frac{1}{\sqrt{(x-a)^2}}-\frac{(x-a)^2}{\left(\sqrt{(x-a)^2}\right)^3}\right]=\begin{cases}0, & x\neq a,\\ \text{indeterminate}, & x=a,\end{cases} \tag{3a}$$

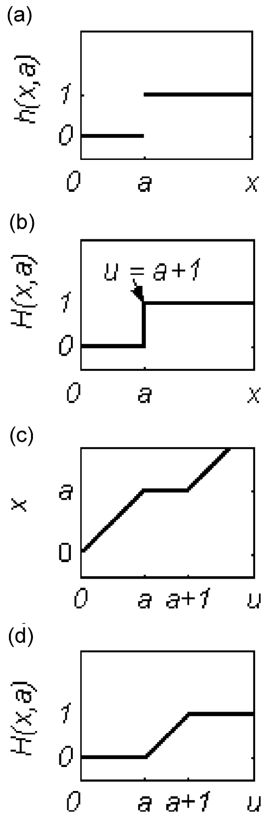

Fig. 1. (a) Riserless Heaviside unit step, direct plot of Eq. (2a), but it represents Eqs. (1), (2a) or (2b), equally well. (b–d) Heaviside unit step with a riser. (b) Direct plot of parametric Eq. (6b) versus (6a). (c) The independent variable versus the parameter u, a direct plot of Eq. (6a). (d) The unit step versus the parameter u, a direct plot of Eq. (6b).

$$\frac{\mathrm{d}h(x,a)}{\mathrm{d}x} = \frac{1}{2}\left[\frac{1}{\sqrt{(x-a)^2}} - \frac{\sqrt{(x-a)^2}}{(x-a)^2}\right] = \begin{cases} 0, & x \neq a, \\ \text{indeterminate}, & x = a. \end{cases} \tag{3b}$$

Thus, it is clear that this process did not yield the Dirac delta. In a way this negative result could have been predicted because, geometrically, the derivative is the slope of the curve of Fig. 1a and there can hardly be any slope associated with the non-existing portion of the curve, i.e., at the "jump "point $x = a$. In architectural terms: this step has no "riser". Incidentally, the results of the differentiation (Eq. (3)) have also established that, when differentiating, the "riserless" unit step should be treated as a constant for $x \neq a$.

3. Heaviside unit step with a riser

3.1. Parametric representation

To remedy this situation the empty vertical portion of the curve may be filled, Fig. 1b. One way of doing this is by resorting to a parametric representation. Note that in this paper a sharp distinction is made between the lower case h and the upper case H. The lower case h is used exclusively to designate the *riserless* unit step of Eqs. (1) and (2) and shown in Fig. 1a. The upper case H is used exclusively in connection with the *parametri*c representation of the unit step, which *does have a riser*, Fig. 1b. The parameter chosen is u, the length of the unit step, in this paper it is measured from the origin but it can be measured from any convenient point in the axis of abcissae. The following equations are established with reference to Fig. 1b:

$$u<a\begin{cases} x_1 = u, \\ H_1 = 0, \end{cases} \quad a<u<a+1\begin{cases} x_2 = a, \\ H_2 = u-a, \end{cases} \quad u>a+1\begin{cases} x_3 = u-1, \\ H_3 = 1. \end{cases} \tag{4}$$

Of course, u must have the same units whether it runs parallel to the horizontal or to the vertical axis or in any other direction. This requirement is complied with by simply handling all the variables involved in dimensionless form.

Eq. (4) for x_j and H_j, $j = 1,2,3$, are each concatenated into a single equation:

$$\begin{aligned} x &= x_1 + h(u,a)(-x_1 + x_2) + h(u,a+1)(-x_2 + x_3), \\ H(x,a) &= H_1 + h(u,a)(-H_1 + H_2) + h(u,a+1)(-H_2 + H_3). \end{aligned} \tag{5}$$

It is convenient to keep in mind that h is used in Eq. (5) as a switch. In the first equation, the first term is x_1. In the second term, due to the action of $h(u,a)$ and the negative sign accompanying it, x_1 is "switched off". In this same second term, and also due to the action of $h(u,a)$ and the positive sign accompanying it, x_2 is simultaneously "switched on". This switching is carried on at the point $u = a$, of course.

Substituting Eq. (4) into Eq. (5) and simplifying yields what are hereby proposed as the parametric equations of the Heaviside unit step with a riser:

$$x = u - h(u,a)(u-a) + h(u,a+1)\{u-(a+1)\}, \tag{6a}$$

$$H(x,a) = h(u,a)(u-a) - h(u,a+1)\{u-(a+1)\}. \tag{6b}$$

Fig. 1b represents $H(x,a)$ versus x and is a direct parametric plot of Eq. (6a) and (6b), Fig. 1c represents x versus u and is a direct plot of Eq. (6a), Fig. 1d represents $H(x,a)$ versus u and is a direct plot of Eq. (6b). (To obtain these graphs a specific value of a was used, of course.)

It is pertinent to point out that even though x is considered to be a *physical independent variable*, both x and $H(x,a)$ are functions of the *geometric independent variable, u*. It is significant that the *single point*, $x = a$, of Fig. 1b has been *expanded* into the *finite interval,* $a<u<a+1$, of Figs. 1c and d.

Note: *Matlab* was used for the plots because it makes a clear distinction between the step with a riser and the step without a riser and so does the TI 92 graphics calculator. A plot of the riserless step, Fig. 1a, in another software or another graphics calculator may result in a trace at the jump point making it indistinguishable from the step with a riser.

4. Proposed parametric equations of the Dirac delta

Differentiating the unit step with a riser

$$\frac{\mathrm{d}H(x,a)}{\mathrm{d}x} = \frac{\mathrm{d}H(x,a)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{7}$$

Differentiating Eq. (6b) and (6a), respectively:

$$\frac{\mathrm{d}H(x,a)}{\mathrm{d}u} = h(u,a) - h(u,a+1), \tag{8}$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 1 - h(u,a) + h(u,a+1). \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (7) yields what is hereby proposed as the parametric representation of the Dirac delta:

$$\delta(x,a) = \frac{h(u,a) - h(u,a+1)}{1 - h(u,a) + h(u,a+1)}, \tag{10a}$$

$$x = u - h(u,a)(u-a) + h(u,a+1)\{u-(a+1)\}. \tag{10b}$$

Obviously, Eq. (10a) and (10b) must be considered simultaneously. Eq. (10b) is the same as Eq. (6a).

4.1. Plots of the proposed representation

The proposed parametric representation of the Dirac delta may be considered to be made up of three component functions of the parameter u, namely, the numerator, $\mathrm{d}H(x,a)/\mathrm{d}u$, the denominator, $\mathrm{d}x/\mathrm{d}u$, and the independent variable, x.

4.1.1. Plots of the component functions versus the parameter

Fig. 2a is a direct plot of the numerator, Eq. (8), Fig. 2b is a direct plot of the denominator, Eq. (9), and Fig. 2c is a direct plot of the independent variable, x, Eq. (10b). All three are functions of the parameter u and have been plotted as such.

4.1.2. Parametric plots

Both the numerator and the denominator of the proposed representation may be termed *displaced point functions* of the independent variable, x. Nonetheless, a plot of these functions is possible within the limitation imposed by the smallest plottable increment (SPI). The SPI in the software used, *Matlab*, is the one that will not leave a trace of the "jump" of the displaced point and, in this case, it is equal to 0.01 while the total length of the abscissa plotted is 3.0, i.e., the gap is 1/300th of the width of the graph. Fig. 3a is the parametric plot of the numerator equation (8) and Fig. 3b is the parametric plot of the denominator equation (9) both of them versus the independent variable x, Eq. (10b). In the original direct plots, the displaced points are clearly there but they are so small that a dot, with a much larger diameter, was printed on top of each one of them to make sure that they will be distinguishable in the final printed copies of this paper. The gaps "left" by the displaced points, which are shown are the original.

But the proposed representation of the delta function may also be plotted within the limitation of the SPI and the understanding that no matter how much the ordinate scale is compressed no displaced point will appear since it is at infinity. Fig. 4 is the direct parametric plot of the Dirac delta, Eq. (10a) versus the independent variable x, Eq. (10b). In this case the vertical scale was compressed to the point, where the maximum value of the ordinate within the plot is, as shown, 10×10^9. The software warns about a division by zero but, nonetheless, it produces the plot.

These plots have also been obtained in the TI-92 graphics calculator, with the advantage that the plotting is witnessed as it proceeds. As in the case of the computer plots, the displaced points are very tenuous. However, it is interesting to see the parametric plot of the Dirac delta proceed up to the beginning of the gap where it

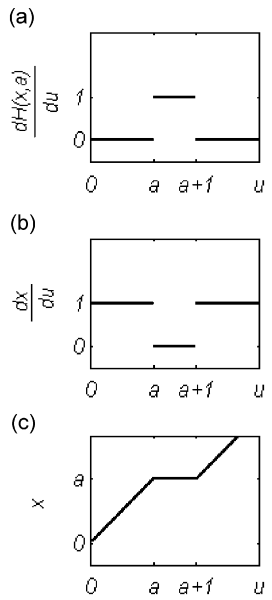

Fig. 2. Plots of the component functions of the Dirac delta versus the parameter u: (a) the numerator, a direct plot of Eq. (8), (b) the denominator, a direct plot of Eq. (9), and (c) the independent variable, a direct plot of Eq. (10b).

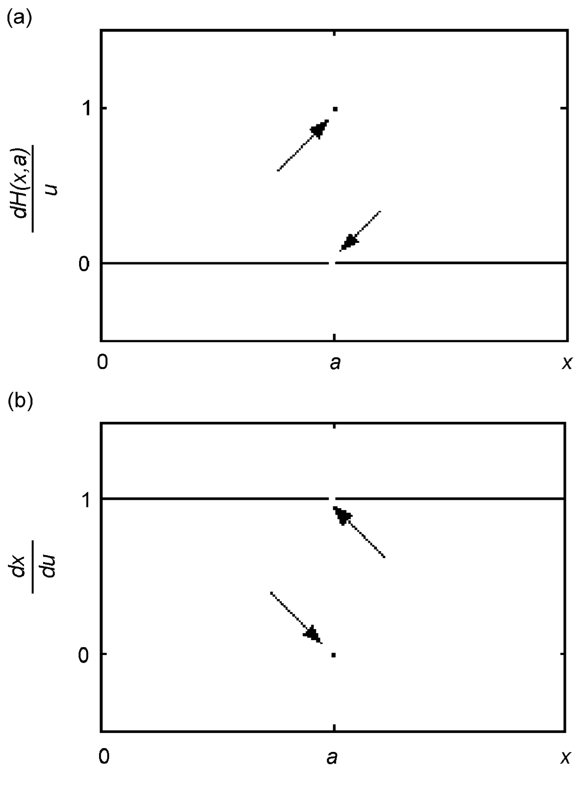

Fig. 3. Plots of the component (displaced point) functions of the Dirac delta versus the independent variable x: (a) the numerator, a direct plot of parametric Eq. (8) versus Eq. (10b), (b) the denominator, a direct plot of parametric Eq. (9) versus Eq. (10b).

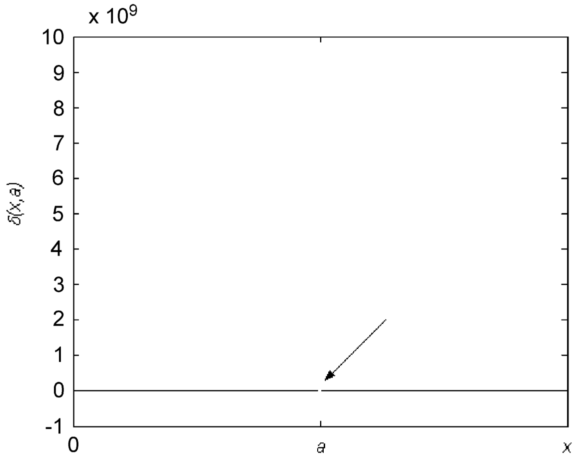

Fig. 4. The proposed Dirac delta function, a direct plot of parametric Eq. (10a) versus Eq. (10b) with a greatly compressed ordinate scale.

stops and, after a considerable length of time, it resumes drawing the graph on the other side of the gap. In the TI-92 plots one unit was added to the function, whether it was the numerator, the denominator or the delta itself, so as to displace the graph away from the axis of abcissae in order to make the gap visible.

Again, it must be pointed out that these results are not obtainable with all software and neither are they obtainable with all graphics calculators.

It is worth noticing that the Dirac delta has been represented by a quotient, Eq. (10a), because this is what makes possible its handling without dealing with infinity directly, as will become apparent in the development of the procedure and the examples. Also, a single point, $x = a$, of Figs. 3 and 4 is represented by a finite interval, $a<u<a+1$, of Fig. 2, and this is what makes possible the handling of the Dirac delta without dealing with infinitesimals directly. These two features will be fully exploited in the following developments.

5. Adequacy of the representation

5.1. Compliance with the definition

Eq. (10) have the values:

$$\begin{aligned} &x = u, && \delta(x,a) = 0, && u<a,\\ &x = a, && \delta(x,a) = \infty, && a<u<a+1,\\ &x = u-1, && \delta(x,a) = 0, && u>a+1. \end{aligned} \tag{11}$$

Thus, Eq. (10) conform to the values of the definition of the Dirac delta.

5.2. Area

The area under the proposed function is

$$A = \int_{-\infty}^{+\infty} \delta(x,a)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \delta(x,a)\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u. \tag{12}$$

Substituting Eq. (9) and (10a) into Eq. (12) and simplifying yields

$$A = \int_{-\infty}^{+\infty} [h(u,a) - h(u,a+1)]\,\mathrm{d}u \tag{13}$$

or equivalently,

$$A = \int_{-\infty}^{a} (0)\,\mathrm{d}u + \int_{a}^{a+1} (1)\,\mathrm{d}u + \int_{a+1}^{+\infty} (0)\,\mathrm{d}u = 1. \tag{14}$$

It is clear that the proposed function has the same area as the Dirac delta.

5.3. Fundamental or "sifting" property

The proposed function will now be tested in connection with the so-called "sifting" property of the Dirac delta:

$$\int_{-\infty}^{+\infty} \delta(x,a) f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \delta(x,a) f(x) \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u. \tag{15}$$

Substituting Eqs. (9) and (10a) into Eq. (15) and simplifying yields

$$\int_{-\infty}^{+\infty} \delta(x,a) f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} [h(u,a) - h(u,a+1)] f(x)\,\mathrm{d}u. \tag{16}$$

But in accordance with Eq. (10b) and Fig. 2c,

$$x = a \quad \text{at} \quad a < u < a+1. \tag{17}$$

Consequently,

$$\int_{-\infty}^{+\infty} \delta(x,a) f(x)\,\mathrm{d}x = \int_{-\infty}^{a} (0)\,\mathrm{d}u + \int_{a}^{a+1} f(a)\,\mathrm{d}u + \int_{a+1}^{\infty} (0)\,\mathrm{d}u, \tag{18}$$

$$\therefore \int_{-\infty}^{+\infty} \delta(x,a) f(x)\,\mathrm{d}x = f(a). \tag{19}$$

Eq. (19) is the fundamental or "sifting" property of the Dirac delta [17], and thus the proposed representation also complies with it.

5.4. Laplace transform

Up to this point, the independent variable has been designated by x assuming that it is dimensionless. However, a procedure to solve impulsive dynamic systems will be presented, where the independent variable is, of course, the time, t. The development of this procedure is based on the proposed parametric representation of the Dirac delta, which requires the independent variable to be *dimensionless*. Furthermore, the t domain of the Laplace transform very often refers to time. In view of this the following dimensionless time will be used:

$$\tau = \frac{t}{\beta}, \quad \tau_0 = \frac{t_0}{\beta}. \tag{20}$$

β has the same units as t and depends on the characteristics of the specific system being considered. Making use of Eq. (20),

$$\delta(t,t_0) = \frac{\mathrm{d}H(t,t_0)}{\mathrm{d}t} = \frac{\mathrm{d}H(t,t_0)}{\beta\,\mathrm{d}\tau}. \tag{21}$$

It is worth emphasizing that $\delta(t,t_0)$ has units of $(\text{time})^{-1}$. Now introducing the parameter u:

$$\delta(t,t_0) = \frac{\mathrm{d}H(t,t_0)}{\beta\,\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}\tau} \tag{22}$$

and converting Eqs. (8)–(10), respectively:

$$\frac{\mathrm{d}H(t,t_0)}{\mathrm{d}u} = h(u,\tau_0) - h(u,\tau_0+1), \tag{23}$$

$$\frac{d\tau}{du} = 1 - h(u, \tau_0) + h(u, \tau_0 + 1), \tag{24}$$

$$\beta\delta(t, t_0) = \frac{h(u, \tau_0) - h(u, \tau_0 + 1)}{1 - h(u, \tau_0) + h(u, \tau_0 + 1)}, \tag{25a}$$

$$\tau = u - h(u, \tau_0)(u - \tau_0) + h(u, \tau_0 + 1)[u - (\tau_0 + 1)]. \tag{25b}$$

Eq. (25) constitute the parametric representation of the dimensionless Dirac delta. The Laplace transform of the Dirac delta is

$$L[\delta(t, t_0)] = \int_0^\infty \delta(t, t_0)e^{-st}\,dt, \tag{26}$$

and according to Eq. (22) and the first of Eq. (20):

$$L[\delta(t, t_0)] = \int_0^\infty \frac{dH(t, t_0)}{\beta\,du}\frac{du}{d\tau}e^{-s\beta\tau}\beta\,d\tau, \tag{27}$$

thus simplifying yields

$$L[\delta(t, t_0)] = \int_0^\infty \frac{dH(t, t_0)}{du}e^{-s\beta\tau}\,du. \tag{28}$$

Substituting Eqs. (23) and (25b) into Eq. (28):

$$L[\delta(t, t_0)] = \int_0^\infty [h(u, \tau_0) - h(u, \tau_0 + 1)]e^{-s\beta\{u - h(u,\tau_0)(u-\tau_0) + h(u,\tau_0+1)[u-(\tau_0+1)]\}}\,du, \tag{29}$$

or equivalently,

$$L[\delta(t, t_0)] = \int_0^{\tau_0} (0)\,du + \int_{\tau_0}^{\tau_0+1} e^{-s\beta\tau_0}\,du + \int_{\tau_0+1}^\infty (0)\,du. \tag{30}$$

Integrating and substituting the second of Eq. (20) into this yields

$$\therefore L[\delta(t, t_0)] = e^{-st_0}, \quad L[\delta(t, 0)] = 1. \tag{31}$$

Thus, Eq. (31) agree with the Laplace transforms given in the tables [18].

6. Development of the procedure for the solution of impulsive systems

The operational procedure will be illustrated in connection with dynamic systems that have an impulsive force as the forcing function:

$$\hat{F} = I\delta(t, 0). \tag{32}$$

As implied by Eq. (32) for the sake of concreteness and simplicity the impulse is considered to occur at

$$t_0 = 0, \quad \therefore \quad \tau_0 = 0. \tag{33}$$

Using this Eqs. (23)–(25) respectively, become

$$\frac{dH(t, 0)}{du} = h(u, 0) - h(u, 1), \tag{34}$$

$$\frac{d\tau}{du} = 1 - h(u, 0) + h(u, 1), \tag{35}$$

$$\beta\delta(t, 0) = \frac{h(u, 0) - h(u, 1)}{1 - h(u, 0) + h(u, 1)}, \tag{36a}$$

$$\tau = u - h(u, 0)u + h(u, 1)(u - 1). \tag{36b}$$

The procedure will be developed for the solution of any impulsive dynamic system which may be represented by the following equation of motion:

$$P\frac{\mathrm{d}^N y}{\mathrm{d}t^N}+f\left(\frac{\mathrm{d}^{N-1}y}{\mathrm{d}t^{N-1}},\frac{\mathrm{d}^{N-2}y}{\mathrm{d}t^{N-2}},\dots,\frac{\mathrm{d}y}{\mathrm{d}t},y\right)=I\delta(t,0)=I\frac{\mathrm{d}H(t,0)}{\mathrm{d}t}, \tag{37}$$

where P is a constant. The function f may be either linear or *nonlinear*. Also, all initial conditions are assumed to be equal to zero.

The dimensionless displacement and the dimensionless time parameters are defined as

$$Y=\frac{y}{\lambda},\quad \tau=\frac{t}{\beta}, \tag{38}$$

where λ and β are constants, which have a form that depends on the particular problem. β has the same units as t and λ has the same units as y. Substituting Eq. (38) into Eq. (37) yields

$$P\frac{\lambda \mathrm{d}^N Y}{\beta^N \mathrm{d}\tau^N}+f_1\left(\frac{\mathrm{d}^{N-1}Y}{\mathrm{d}\tau^{N-1}},\frac{\mathrm{d}^{N-2}Y}{\mathrm{d}\tau^{N-2}},\dots,\frac{\mathrm{d}Y}{\mathrm{d}\tau},Y\right)=I\frac{\mathrm{d}H(t,0)}{\beta\,\mathrm{d}\tau}. \tag{39}$$

Multiplying Eq. (39) by β/I results in the following dimensionless equation:

$$\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}^N Y}{\mathrm{d}\tau^N}+f_2\left(\frac{\mathrm{d}^{N-1}Y}{\mathrm{d}\tau^{N-1}},\frac{\mathrm{d}^{N-2}Y}{\mathrm{d}\tau^{N-2}},\dots,\frac{\mathrm{d}^2Y}{\mathrm{d}\tau^2},\frac{\mathrm{d}Y}{\mathrm{d}\tau},Y\right)=\frac{\mathrm{d}H(t,0)}{\mathrm{d}\tau}. \tag{40}$$

Eq. (40) is replaced by the equivalent system of N state equations:

$$\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}Y_{N-1}}{\mathrm{d}\tau}+f_2(Y_{N-1},Y_{N-2},\dots,Y_2,Y_1,Y)=\frac{\mathrm{d}H(t,0)}{\mathrm{d}\tau},$$

$$\frac{\mathrm{d}Y_{N-2}}{\mathrm{d}\tau}=Y_{N-1},\ \frac{\mathrm{d}Y_{N-3}}{\mathrm{d}\tau}=Y_{N-2},\dots,\frac{\mathrm{d}Y_1}{\mathrm{d}\tau}=Y_2,\ \frac{\mathrm{d}Y}{\mathrm{d}\tau}=Y_1. \tag{41}$$

The parameter u is introduced as follows:

$$\begin{gathered}\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}Y_{N-1}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\tau}+f_2(Y_{N-1},Y_{N-2},\dots,Y_2,Y_1,Y)=\frac{\mathrm{d}H(t,0)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\tau},\\ \frac{\mathrm{d}Y_{N-2}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\tau}=Y_{N-1},\quad \frac{\mathrm{d}Y_{N-3}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\tau}=Y_{N-2},\dots,\frac{\mathrm{d}Y_1}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\tau}=Y_2,\quad \frac{\mathrm{d}Y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\tau}=Y_1,\end{gathered} \tag{42}$$

and these equations are multiplied by $\mathrm{d}\tau/\mathrm{d}u$ to yield

$$\begin{gathered}\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}Y_{N-1}}{\mathrm{d}u}+\frac{\mathrm{d}\tau}{\mathrm{d}u}f_2(Y_{N-1},Y_{N-2},\dots,Y_2,Y_1,Y)=\frac{\mathrm{d}H(t,0)}{\mathrm{d}u},\\ \frac{\mathrm{d}Y_{N-2}}{\mathrm{d}u}=\frac{\mathrm{d}\tau}{\mathrm{d}u}Y_{N-1},\quad \frac{\mathrm{d}Y_{N-3}}{\mathrm{d}u}=\frac{\mathrm{d}\tau}{\mathrm{d}u}Y_{N-2},\dots,\frac{\mathrm{d}Y_1}{\mathrm{d}u}=\frac{\mathrm{d}\tau}{\mathrm{d}u}Y_2,\quad \frac{\mathrm{d}Y}{\mathrm{d}u}=\frac{\mathrm{d}\tau}{\mathrm{d}u}Y_1.\end{gathered} \tag{43}$$

Substituting Eqs. (34) and (35) into Eq. (43),

$$\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}Y_{N-1}}{\mathrm{d}u}+[1-h(u,0)+h(u,1)]f_2(Y_{N-1},Y_{N-2},\dots,Y_2,Y_1,Y)=h(u,0)-h(u,1),$$

$$\frac{\mathrm{d}Y_{N-2}}{\mathrm{d}u}=[1-h(u,0)+h(u,1)]Y_{N-1},\quad \frac{\mathrm{d}Y_{N-3}}{\mathrm{d}u}=[1-h(u,0)+h(u,1)]Y_{N-2},\dots,$$

$$\frac{\mathrm{d}Y_1}{\mathrm{d}u}=[1-h(u,0)+h(u,1)]Y_2,\quad \frac{\mathrm{d}Y}{\mathrm{d}u}=[1-h(u,0)+h(u,1)]Y_1. \tag{44}$$

During the impulse instant, i.e., when $0<u<1$ and $\tau=0$ Eq. (44) becomes

$$\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}Y_{N-1,i}}{\mathrm{d}u}=1,$$
$$\frac{\mathrm{d}Y_{N-2,i}}{\mathrm{d}u}=0,\quad \frac{\mathrm{d}Y_{N-3,i}}{\mathrm{d}u}=0,\ldots,\frac{\mathrm{d}Y_{1i}}{\mathrm{d}u}=0,\quad \frac{\mathrm{d}Y_{i}}{\mathrm{d}u}=0. \tag{45}$$

Integrating these equations yields

$$Y_{N-1,i}=\left(\frac{I\beta^{N-1}}{P\lambda}\right)u+C_{N-1},$$
$$Y_{N-2,i}=C_{N-2},\quad Y_{N-3,i}=C_{N-3},\ldots,Y_{1i}=C_1,\quad Y_i=C. \tag{46}$$

The initial conditions are all equal to zero at the "beginning" of the impulse instant, i.e., $u=0$, ($\tau=0$ and $t=0$), consequently all the constants of integration are also equal to zero so Eq. (46) becomes

$$Y_{N-1,i}=\left(\frac{I\beta^{N-1}}{P\lambda}\right)u,$$
$$Y_{N-2,i}=0,\quad Y_{N-3,i}=0,\ldots,Y_{1i}=0,\quad Y_i=0. \tag{47}$$

6.1. Parametric solution

At the "end" of the impulse instant, $u=1$ ($\tau=0$, and $t=0$), and beginning of post-impulse time:

$$Y_{N-1,p}\big|_{u=1,\ \tau=0}=\frac{I\beta^{N-1}}{P\lambda},$$
$$Y_{N-2,p}\big|_{u=1,\ \tau=0},\quad Y_{N-3,p}\big|_{u=1,\ \tau=0}=0,\ldots,Y_{1p}\big|_{u=1,\ \tau=0}=0,\quad Y_{p}\big|_{u=1,\ \tau=0}=0. \tag{48}$$

Eq. (48) is the initial conditions of post-impulse time.

Substituting $u>1$ ($\tau>0$ and $t>0$) into the first of Eq. (44) yields the equation of post-impulse time:

$$\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}Y_{N-1,p}}{\mathrm{d}u}+f_2\left(Y_{N-1,p},\quad Y_{N-2,p},\ldots,Y_{1p},\quad Y_p\right)=0,$$

$$\frac{\mathrm{d}Y_{N-2,p}}{\mathrm{d}u}=Y_{N-1,p},\quad \frac{\mathrm{d}Y_{N-3,p}}{\mathrm{d}u}=Y_{N-2,p},\ldots,\frac{\mathrm{d}Y_{1,p}}{\mathrm{d}u}=Y_{2,p},\quad \frac{\mathrm{d}Y_{p}}{\mathrm{d}u}=Y_{1,p}$$

or

$$\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}^{N-1}Y_p}{\mathrm{d}\tau^{N-1}}\right)+f_2\left(\frac{\mathrm{d}^{N-1}Y_p}{\mathrm{d}\tau^{N-1}},\frac{\mathrm{d}^{N-2}Y_p}{\mathrm{d}\tau^{N-2}},\ldots,\frac{\mathrm{d}Y_p}{\mathrm{d}\tau},Y_p\right)=0$$
$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{d^{N-2}Y_p}{d\tau^{N-2}}\right)=Y_{N-1,p},\quad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}^{N-3}Y_p}{\mathrm{d}\tau^{N-3}}\right)=Y_{N-2,p},\ldots,\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}Y_{1,p}}{\mathrm{d}\tau}\right)=Y_{2,p},\quad \frac{\mathrm{d}Y_p}{\mathrm{d}u}=Y_{1,p}, \tag{49}$$

but

$$\frac{\mathrm{d}}{\mathrm{d}u}=\frac{\mathrm{d}\tau}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{50}$$

According to Eq. (36b) at post-impulse time,

$$u>1,\quad \tau=u-1 \tag{51}$$

and according to Eq. (51),

$$u>1,\quad \frac{\mathrm{d}\tau}{\mathrm{d}u}=1. \tag{52}$$

Substituting Eq. (52) into Eq. (50) results in

$$u>1, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}=\frac{\mathrm{d}}{\mathrm{d}u} \tag{53}$$

and, in view of Eq. (53), Eq. (49) are the equivalent of the following single equation:

$$\left(\frac{P\lambda}{I\beta^{N-1}}\right)\frac{\mathrm{d}^N Y_p}{\mathrm{d}u^N}+f_2\left(\frac{\mathrm{d}^{N-1} Y_p}{\mathrm{d}u^{N-1}},\ \frac{\mathrm{d}^{N-2} Y_p}{\mathrm{d}u^{N-2}},\dots,\frac{\mathrm{d}Y_p}{\mathrm{d}u},\ Y_p\right)=0. \tag{54}$$

The post-impulse displacement $Y_p(u)$ may be obtained by solving Eq. (54) subject to the post-impulse initial conditions, Eq. (48). $Y_i(u)$ is the last of Eq. (47). Proceeding as in Eq. (5) the complete parametric solution is

$$Y(u)=h(u,0)\,Y_i(u)+h(u,1)\left\{-Y_i(u)+Y_p(u)\right\}. \tag{55}$$

Obviously, in the case of linear problems, the equation of the impulse instant is related to the particular integral and the post-impulse equation is the homogeneous equation.

6.2. Direct solution

According to the definition of the numerical values of the Dirac delta, at post-impulse time:

$$t>0, \quad \delta(t,0)=0. \tag{56}$$

Substituting Eq. (56) into Eq. (37) yields the post-impulse equation

$$t>0, \quad P\frac{\mathrm{d}^N y_p}{\mathrm{d}t^N}+f\left(\frac{\mathrm{d}^{N-1} y_p}{\mathrm{d}t^{N-1}},\frac{\mathrm{d}^{N-2} y_p}{\mathrm{d}t^{N-2}},\dots,\frac{\mathrm{d}y_p}{\mathrm{d}t},y_p\right)=0. \tag{57}$$

Substituting Eq. (38) into Eq. (48) yields the post-impulse initial conditions in a form appropriate for this solution

$$\begin{gathered}\left.\frac{\mathrm{d}^{N-1} y_p}{\mathrm{d}t^{N-1}}\right|_{t=0}=\frac{I}{P},\\ \left.\frac{\mathrm{d}^{N-2} y_p}{\mathrm{d}t^{N-2}}\right|_{t=0}=0, \quad \left.\frac{\mathrm{d}^{N-3} y_p}{\mathrm{d}t^{N-3}}\right|_{t=0}=0,\dots,\left.\frac{\mathrm{d}y_p}{\mathrm{d}t}\right|_{t=0}=0, \quad y_p(0)=0.\end{gathered} \tag{58}$$

Solving the post-impulse Eq. (57) subject to the post-impulse initial conditions (58) yields the direct solution

$$y=h(t,0)y_p(t). \tag{59}$$

The first factor, $h(t{,}0)$, is included because there is no process for $t<0$.

Note: In the development of the procedures, in order to neither lose sight of the basic idea nor complicate the notation, all the initial conditions were assumed to be zero. If they are not, they are simply added, of course.

7. The direct procedure in words

Although both solutions are based on the proposed parametric representation, it is quite clear that the direct solution is much easier to obtain than the parametric solution, among other things, because it does not require conversion to dimensionless quantities. Indeed the direct procedure is so simple that it is well worth putting it in words.

The characteristics of the problem that it is possible to solve with the procedure are: the system can be modeled by and Nth-order differential equation of motion with an impulse excitation, either linear or with nonlinearities in the terms containing derivatives of order less than N, and all initial conditions equal to zero.

Procedure: Solve the post-impulse equation, i.e., the original differential equation of motion with zero excitation, for the following post-impulse initial conditions: for the order $N-1$ derivative, the magnitude of the impulse divided by the coefficient of the term containing the Nth-order derivative; and for all the rest, zero. See "note" at the end of Section 6.2.

8. Examples of solutions

The following solutions are of an elementary nature. However, they are presented here to reinforce the arguments referring to the adequacy of the representation, and to illustrate both the application and the validity of the procedure.

8.1. Example of a linear, 3rd-order system

In order to illustrate the solution of a higher-order system, this example referring to a mechanical system includes feedback control.

Fig. 5 is the block diagram of an "industry standard" position control system [19]. An external step disturbance torque becomes an impulse torque internally. The displacement $\theta(t)$ will be obtained (in terms of the roots of the characteristic equation) for the reference position, $\theta_r = 0$.

Nomenclature used in Section 8.1 is as follows:

μ rotational viscous friction coefficient, N m s
J moment of inertia of motor rotor and load, N m s^2
K position proportional gain, s^{-1}
K_i velocity integral gain, V
$K_m = K_T/R_a$ motor gain (armature inductance neglected [20]), N m V^{-1}
K_p velocity proportional gain, V s
K_T torque constant of motor, N m A^{-1}
R_a armature resistance of motor, Ω
$T_D = T_d\, H(t,0)$ step disturbance torque, N m
T_d magnitude of step disturbance torque, N m
T_m motor torque, N m
V motor voltage, V
θ controlled position, rad
θ_r reference position, rad
ω output velocity, rad s^{-1}
ω_i input velocity, rad s^{-1}

From the block diagram, the differential equation of motion is established thus,

$$J\frac{\mathrm{d}^3\theta}{\mathrm{d}t^3} + (\mu + K_m K_p)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + (K_m K_p K + K_m K_i)\frac{\mathrm{d}\theta}{\mathrm{d}t} + K_m K_i K\theta = -\frac{\mathrm{d}T_D}{\mathrm{d}t}. \tag{60}$$

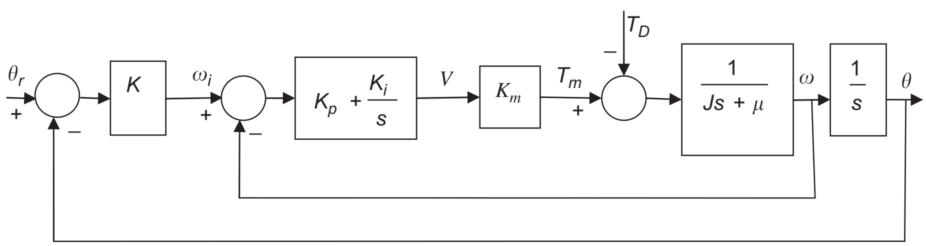

Fig. 5. Example from Section 8.1. Block diagram of a position control system [19].

The step disturbance torque may be expressed as

$$T_D = T_d H(t,0) \tag{61}$$

substituting into the equation of motion:

$$J\frac{d^3\theta}{dt^3} + (\mu + K_m K_p)\frac{d^2\theta}{dt^2} + (K_m K_p K + K_m K_i)\frac{d\theta}{dt} + K_m K_i K\theta = -T_d\delta(t,0), \tag{62}$$

the order of the equation is

$$N = 3, \tag{63}$$

the post-impulse equation is Eq. (62) with zero excitation:

$$J\frac{d^3\theta}{dt^3} + (\mu + K_m K_p)\frac{d^2\theta}{dt^2} + (K_m K_p K + K_m K_i)\frac{d\theta}{dt} + K_m K_i K\theta = 0, \tag{64}$$

the post-impulse initial conditions are for the order $N-1 = 2$ derivative:

$$\ddot{\theta}(0) = \frac{\text{magnitude of the impulse}}{\text{coefficient of the } N\text{th order term}} = -\frac{T_d}{J}, \tag{65a}$$

for the order $N-2 = 1$ and $N-3 = 0$ derivatives, respectively:

$$\dot{\theta}(0) = 0,$$
$$\theta(0) = 0. \tag{65b}$$

Eq. (64) is to be solved subject to initial conditions, Eq. (65). The form of the solution depends on the numerical values of the coefficients, but, in terms of the roots of the characteristic equation: r_1, r_2 and r_3, the displacement is

$$\theta = h(t,0)\frac{T_d}{J}\frac{(r_2 - r_3)e^{-r_1 t} + (r_3 - r_1)e^{-r_2 t} + (r_1 - r_2)e^{-r_3 t}}{r_1^2(r_3 - r_2) + r_2^2(r_1 - r_3) + r_3^2(r_2 - r_1)}, \tag{66}$$

which is equivalent to the solution obtained by use of the Laplace transform, i.e.:

$$\theta = -h(t,0)\frac{T_d}{J}\left[\frac{e^{-r_1 t}}{(r_2 - r_1)(r_3 - r_1)} + \frac{e^{-r_2 t}}{(r_1 - r_2)(r_3 - r_2)} + \frac{e^{-r_3 t}}{(r_1 - r_3)(r_2 - r_3)}\right]. \tag{67}$$

8.2. Example of nonlinear, 2nd-order system

The response of a large oscillation pendulum subjected to a linear impulse of magnitude I will be obtained in the form of the velocity, v as a function of the height, y, of the oscillating mass. Referring to Fig. 6, the equation of motion is

$$mL\frac{d^2\theta}{dt^2} + mg\sin\theta = I\delta(t,0), \tag{68}$$

the order of the equation is

$$N = 2. \tag{69}$$

The post-impulse equation is Eq. (68) with zero excitation:

$$L\frac{d^2\theta}{dt^2} + g\sin\theta = 0 \tag{70}$$

or

$$L\omega\frac{d\omega}{d\theta} + g\sin\theta = 0. \tag{71}$$

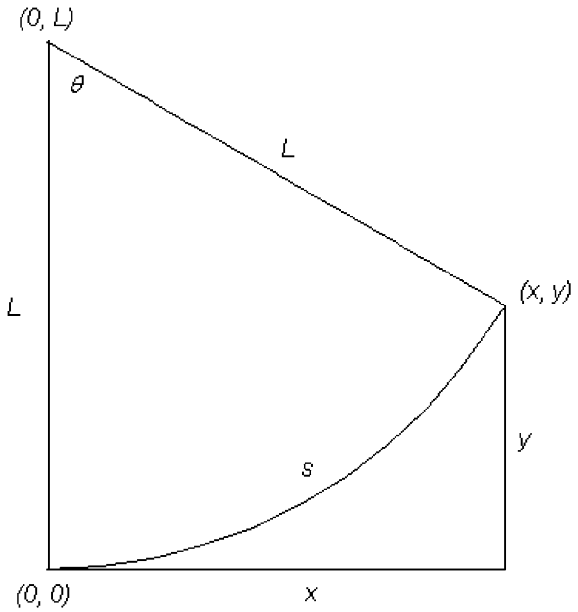

Fig. 6. Example from Section 8.2. Geometry of the large oscillation pendulum, point $(0, L)$ is the positon of the pivot point, point $(0, 0)$ is the equilibrium position of the center of mass, and point (x, y) is the instantaneous position of the center of mass.

The post-impulse initial conditions are for the order $N-1=1$ derivative:

$$\dot{\theta}(0)=\omega(0)=\frac{\text{magnitude of the impulse}}{\text{coefficient of the } N\text{th order term}}=\frac{I}{mL} \tag{72a}$$

and for the order $N-2=0$ derivative

$$\theta(0)=0. \tag{72b}$$

Integrating Eq. (71) subject to the initial conditions of Eq. (72) results in

$$\omega=\pm\sqrt{\frac{2g}{L}(\cos\,\theta-1)+\frac{I^2}{m^2L^2}} \tag{73}$$

multiplying by L,

$$v=\pm\sqrt{2gL(\cos\,\theta-1)+\frac{I^2}{m^2}}. \tag{74}$$

But in accordance with Fig. 6

$$y=L(1-\cos\,\theta), \tag{75}$$

$$\therefore \quad v=\pm\sqrt{\frac{I^2}{m^2}-2gy}. \tag{76}$$

This relation was obtained without the need of any dimensionless quantities. However, in order for the plot to represent a general solution, it is convenient to convert Eq. (76) into its dimensionless form

$$\frac{m}{I}v=\pm\sqrt{1-2\frac{gm^2}{I^2}y} \tag{77a}$$

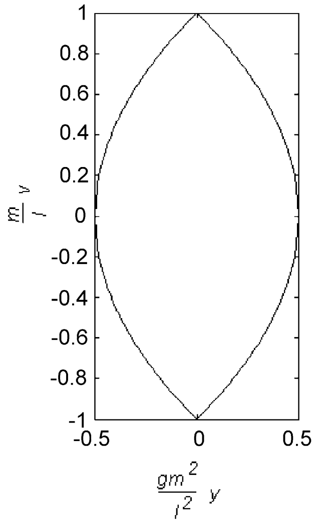

Fig. 7. Example from Section 8.2. Velocity versus displacement for the large oscillation pendulum subjected to an impulse of magnitude I, a plot of Eq. (77).

but symmetry requires that additionally

$$\frac{m}{I}v = \pm\sqrt{1 + 2\frac{gm^2}{I^2}y}, \tag{77b}$$

see Fig. 7.

9. Discussion

As is apparent from the development of the procedure and the examples, the proposed representation of the Dirac delta, when applied to the solution of an impulsive vibrating system, separates the equation of motion into two distinct equations: the impulse-instant equation and the post-impulse time equation. No time flows during the impulse instant, but changes take place and it is convenient to express them in terms of a variable. Parameterization provides such a variable: the parameter u. Furthermore, the post-impulse equation may be expressed in terms of the same variable. This provides a link between the two equations, i.e., the condition of u and its derivatives during the transition or interphase is common to both equations.

Eq. (47) formally establish that the physical components associated with any derivative of order less than N, the order of the differential equation of motion, do not participate in the dynamic process that takes place during the impulse instant and, consequently they do not have any effect on the initial conditions of the post-impulse equation. In the case of the control system example in Section 8.1 only the combined moment of inertia of the motor rotor and the load has this effect, but neither the motor gain, nor the controllers, nor the rotational friction do. In the case of the large oscillation pendulum example in Section 8.2, again, only the mass of the pendulum does have the effect, but the weight does not. However, all of the elements in both systems of the examples participate in the post-impulse process.

Problems relating to linear impulsive vibrating systems may be solved by use of the Laplace transform but this requires going from the t to the s domain and back and, sometimes, it is necessary to deal with partial fractions. None of that is required with the proposed procedure. As is well known, no nonlinear problem may be solved by the Laplace transform method.

A large class of nonlinear problems may be solved by the proposed method but the nonlinearities must be dealt with by using the usual methods. However, the nonlinearities are present only in the post-impulse equation which has a right member equal to zero. This feature may simplify matters as it did in the example in Section 8.2, where the post-impulse equation is indeed nonlinear but, because its right member is equal to zero, it is separable.

10. Conclusions

A very simple concept, i.e., a riser in connection with the Heaviside unit step function made it possible to carry out a conventional differentiation of this function. The introduction of the riser required a parametric representation of the unit step. The resulting derivative, in the form of a pair of parametric equations, has been proposed herein to represent the Dirac delta. The proposed delta representation has been shown to possess the same functional values and the same area under the curve as those specified in the definition, it also complies with the sifting property. Furthermore, the Laplace transform obtained from this representation coincides with that given in the tables in its most general form.

Using this parametric representation, a rational and very simple procedure for the solution of a wide variety of dynamic impulsive systems has also been presented. This procedure is applicable to linear problems but also to a large class of nonlinear problems, i.e., when in the differential equation of motion the nonlinearities are present in the terms containing the displacement or any time derivative of one order less than that which characterizes the differential equation itself.

References

[1] E.W. Weisstein, *Delta function, MathWorld—A Wolfram Web Resource* ⟨http://mathworld.wolfram.com/DeltaFunction.html⟩.

[2] N. Drakos, *Delta functions, Computer Based Learning Unit*, University of Leeds, 1994 ⟨http://musr.physics.ubc.ca/⟩

[3] J. Spanier, K.B. Oldham, *An Atlas of Functions*, Hemisphere, New York, 1987, pp. 79–81.

[4] H.G. Weber, G.B. Arfken, *Essential Mathematical Methods for Physicists*, Academic Press, San Diego, 2004, pp. 86–90.

[5] G.G. Walter, Approximation of the delta function by wavelets, *Journal of Approximation Theory* 71 (1992) 329–343.

[6] B. Schomburg, On the approximation of the delta distributions in Sobolev spaces of negative order, *Applied Analysis* 36 (1990) 89–93.

[7] R.N. Bracewell, *The Fourier Transform and its Applications*, third ed., McGraw-Hill, Boston, 2000, p. 5.

[8] C. Lanczos, *Applied Analysis*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1961, pp. 222, 262.

[9] M.D. Greenberg, *Advanced Engineering Mathematics*, second ed., Prentice-Hall, Upper Saddle River, NJ, 1998, pp. 275, 276.

[10] L. Meirovitch, *Elements of Vibration Analysis*, McGraw-Hill, Kogakusha, 1975 International student edition, pp. 65–67.

[11] R.F. Hoskins, *Generalized Functions*, Ellis Horwood Ltd., Wiley, Chichester, W. Sussex, England, 1979, p. 42.

[12] E. Chicurel-Uziel, Non-piecewise representation of discontinuous functions and its application to the Clebsch method for beam deflections, *Meccanica* 34 (4) (1999/2000) 281–285.

[13] E. Chicurel-Uziel, Closed-form solution for response of linear systems subject to periodic, non-harmonic excitation, *Journal of Multi-body Dynamics* 214 (K3) (2000) 189–193.

[14] E. Chicurel-Uziel, Exact, no transform solution of multi degree-of-freedom vibrations with non-harmonic, periodic excitation, *Journal of Multi-body Dynamics* 215 (K3) (2001) 163–169.

[15] E. Chicurel-Uziel, Exact, single equation, closed form solution of vibrating systems with piecewise linear springs, *Journal of Sound and Vibration* 245 (2) (2001) 285–301.

[16] E. Chicurel-Uziel, Single equation without inequalities to represent a composite curve, *Computer-aided Geometric Design* 21 (1) (2003/2004) 23–42.

[17] W.B. Roos, *Analytic Functions and Distributions in Physics and Engineering*, Wiley, New York, 1969, p. 291.

[18] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products*, fifth ed., Academic Press, New York, 1994, p. 1182.

[19] H. Tan, J. Chang, M.A. Chaffee, Practical motion control and PI design, *Proceedings of the American Control Conference*, Vol. 1(6), Chicago, IL, 2000, pp. 529–533.

[20] B.C. Kuo, *Automatic Control Systems*, seventh ed., Prentice-Hall, Englewood Cliffs, NJ, 1995, p. 404.